

# More on triangular mass matrices for fermions

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## Abstract

A direct proof is given here which shows that instead of 6 complex numbers, the triangular mass matrix for each sector could just be expressed in terms of 5 by performing a specific weak basis transformation, leading therefore to a new textures for triangular mass matrices. Furthermore, starting with the set of 20 real parameters for both sectors, it can shown that 6 redundant parameters can be removed by using the rephasing freedom.

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One of the major unsolved problem in particle physics is to understand flavor mixing and fermion masses, which are free parameters in the standard model.

Many speculations have been made and some Ansätze were proposed by introducing extra symmetries to cast the mass matrices in some particular forms [1, 2, 3, 4].

A popular Ansatz suggested by Fritzsch is the Nearest-Neighbor Interactions ( NNI ) [1] which has been for a long time a mysterious parametrization. Branco et al. [7] have been the first to show that for non-hermitian mass matrices some textures à la Fritzsch were just a rewriting of the mass matrices in a special basis without any loss of generality by using this basis.

Making use of the freedom in choosing the right-handed bases for the fermion fields, a class of triangular mass matrices has been introduced recently. It was shown, in particular, that the effective triangular mass matrix  $\mathcal{T}_{eff}$ , which is obtained by shifting the diagonalization in one sector, is reconstructed analytically from the moduli of the CKM matrix and quark masses. This reconstruction is unique up to trivial phase redefinitions.

Such patterns arise in the framework of Marseille–Mainz noncommutative geometry [5], namely triangular mass matrices [6, 10]. They are typical for reducible but indecomposable representations of graded Lie algebras,

$$\begin{pmatrix} t_{11} & 0 & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & t_{33} \end{pmatrix}, \quad \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix} \quad (1)$$

where  $t_{11}, t_{21}, t_{22}, t_{31}, t_{32}, t_{33}$  for lower-triangular mass matrix ( resp.  $t_{11}, t_{12}, t_{13}, t_{22}, t_{23}, t_{33}$  for upper-triangular mass matrix ) are 6 complex numbers. Of course, not all of these parameters are physically relevant, since some of them can be made real by performing a suitable phase transformation on the fermion fields.

In a recent paper [8], see also [9], an indirect proof was given that there is a weak basis transformation that reduces the number of complex parameters to 5 for each sector. We have also made a bridge and a close connection between the NNI and triangular mass matrices.

Here, we present a direct proof to express these triangular mass matrices in a economic and concise way with 5 parameters, through a specific weak basis transformation which mean that the extra parameter is either zero or dependent of the others.

In fact, it can be shown that from the set of 20 real parameters for both sectors, 6 phases can be removed by exploiting the freedom in redefining the fermion fields leaving therefore 5 real moduli and two real phases for each sector. Moreover one sector can be made completely real with 5 real moduli and consequently the other sector has 5 real moduli and two real relative phases.

It is well known, within the standard model that the two sets of mass matrices  $(M_u, M_d)$  and  $(\mathcal{M}_u, \mathcal{M}_d)$  related to each other through,

$$M_u = U^\dagger \mathcal{M}_u V_u, \quad M_d = U^\dagger \mathcal{M}_d V_d \quad (2)$$

give rise to the same physics ( i.e. same masses and mixings ).

We have then the following two propositions for triangular mass matrices,

Theorem1 : ( for two lower triangular mass matrices )

Given any two nonsingular  $3 \times 3$  lower triangular mass matrices  $\mathcal{T}_u, \mathcal{T}_d$ , there exists always a weak basis transformation such that the new mass matrices  $T_u, T_d$  are lower triangular with vanishing matrix elements (2,1).

Proof :

we first prove that the transformed mass matrices are lower triangular. Indeed,  $\mathcal{T}_{u,d}$  transform as  $U^\dagger \mathcal{T}_{u,d} V_{u,d}$ . It is clear that, for a fixed unitary  $U$ , the unitary matrices  $V_{u,d}$  could always be chosen such that  $T_{u,d} = U^\dagger \mathcal{T}_{u,d} V_{u,d}$  are lower triangular.

To prove the second part, i.e  $(T_{u,d})_{21} = 0$ , just consider the hermitian mass matrices  $\mathcal{H}_u = \mathcal{T}_u \mathcal{T}_u^\dagger$  and  $\mathcal{H}_d = \mathcal{T}_d \mathcal{T}_d^\dagger$  which transform as  $H_{u,d} = U^\dagger \mathcal{H}_{u,d} U$  with :

$$(H_u)_{21} = (H_d)_{21} = 0. \quad (3)$$

and construct the unitary matrix  $U$  such that :

$$U_{i2}^* (\mathcal{H}_u)_{ij} U_{j1} = 0$$

$$\begin{aligned} U_{i2}^*(\mathcal{H}_d)_{ij}U_{j1} &= 0 \\ U_{i2}^*U_{i1} &= 0 \end{aligned} \quad (4)$$

which means that the vector  $U_{i2}^*$  is orthogonal to the three vectors  $(\mathcal{H}_u)_{ij}U_{j1}$ ,  $(\mathcal{H}_d)_{ij}U_{j1}$  and  $U_{i1}$ .

For this to be possible, choose  $U_{i1}$  such that these three vectors are linearly dependent, i.e.

$$aU_{i1} + b(\mathcal{H}_u)_{ij}U_{j1} + c(\mathcal{H}_d)_{ij}U_{j1} = 0 \quad (5)$$

where  $a, b$  and  $c$  are some nonvanishing parameters.

Therefore  $U_{i1}$  is just a normalized eigenvector of the matrix  $\mathcal{H}_u + \frac{c}{b}\mathcal{H}_d$ , with eigenvalue  $-\frac{a}{b}$ .

Starting from  $U_{i1}$ , the vector  $U_{i2}$  satisfying (4) is constructed as follows :

$$U_{i2} = N\epsilon_{ijk}U_{j1}^*U_{l1}(\mathcal{H}_u)_{lk} \quad (6)$$

Once  $U_{i1}$  and  $U_{i2}$  are found, therefore the full matrix  $U$  can be constructed.

Now, since the new  $3 \times 3$  hermitian matrices have vanishing (2,1), i.e.

$$(H_{u,d})_{21} = (T_{u,d})_{2i}(T_{u,d}^\dagger)_{i1} = 0$$

it follows that,

$$(t_{u,d})_{21}(t_{u,d}^*)_{11} = 0$$

where we have used the fact that  $T_{u,d}$  is lower triangular.

From the nonsingularity of  $T_{u,d}$ , it implies that :

$$(t_{u,d})_{21} = 0 \quad (7)$$

This completes the proof.

**Theorem 2 :** ( for two upper triangular mass matrices )

Given any two nonsingular  $3 \times 3$  upper triangular mass matrices  $\mathcal{T}_u, \mathcal{T}_d$ , there is always a weak basis transformation such that the new mass matrices  $T_u, T_d$  are upper triangular with the matrix elements satisfying,  $(t_{u,d})_{12}(t_{u,d}^*)_{22} + (t_{u,d})_{13}(t_{u,d}^*)_{23} = 0$ .

**Proof :**

It is always possible to choose the transformed mass matrices  $T_{u,d}$  upper triangular with vanishing matrix element  $(H_{u,d})_{12}$  for the hermitian matrix, see above.

Now, the requirement  $(H_{u,d})_{12} = 0$  leads to :

$$(T_{u,d})_{1i}(T_{u,d}^\dagger)_{i2} = 0$$

and since  $T_{u,d}$  are upper triangular, we get :

$$(t_{u,d})_{12}(t_{u,d}^*)_{22} + (t_{u,d})_{13}(t_{u,d}^*)_{23} = 0 \quad (8)$$

End of the proof.

Now the following remarks are in order. Instead of choosing for both sectors, upper or lower triangular mass matrices, we could choose for one sector lower triangular and for the other sector upper triangular. In this case, we get for the sector with lower one a vanishing matrix element (2,1) and for the upper a relation between matrix elements similar to (8).

It is clear that we can also find a weak basis such that the lower triangular mass matrices  $T_u, T_d$  are transformed into lower triangular mass matrices with vanishing matrix elements (3,1) or (3,2), see Appendix here or [8].

Similarly for upper triangular, the transformed matrix is upper triangular with one matrix element written in terms of the others.

## Appendix:

Here, we list all the textures of these new triangular mass matrices where we have dropped the flavor indices  $u, d$ .

**I-weak basis such that  $H_{12} = H_{21} = 0$  ,**

1-lower triangular :

a)  $t_{21} = 0$ ,

$$T_b = \begin{pmatrix} t_{11} & 0 & 0 \\ 0 & t_{22} & 0 \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$$

b) singular matrix with  $t_{11} = 0$ ,

$$T_b = \begin{pmatrix} 0 & 0 & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$$

2-upper triangular :

$$T_h = \begin{pmatrix} t_{11} & -\frac{t_{22}t_{23}^*t_{13}}{|t_{22}|^2} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix}$$

**II-weak basis such that  $H_{23} = H_{32} = 0$ ,**

1-lower triangular :

$$T_b = \begin{pmatrix} t_{11} & 0 & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & -\frac{t_{22}t_{21}^*t_{31}}{|t_{22}|^2} & t_{33} \end{pmatrix}$$

2- upper-triangular matrix :

a)  $t_{23} = 0$ ,

$$T_h = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix}$$

b) singular matrix with  $t_{33} = 0$ ,

$$T_h = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

**III–weak basis such that  $H_{13} = H_{31} = 0$  ,**

1–lower triangular :

a)  $t_{31} = 0$ ,

$$T_b = \begin{pmatrix} t_{11} & 0 & 0 \\ t_{21} & t_{22} & 0 \\ 0 & t_{23} & t_{33} \end{pmatrix}$$

b) singular matrix for  $t_{11} = 0$ ,

$$T_b = \begin{pmatrix} 0 & 0 & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$$

2–upper triangular :

a)  $t_{13} = 0$ ,

$$T_h = \begin{pmatrix} t_{11} & t_{12} & 0 \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix}$$

b) singular matrix for  $t_{33} = 0$ ,

$$T_b = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

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